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Bytsenko, A. A.

2019-01

Bytsenko , A A , Chaichian , M & Goncalves , A E 2019 , ' The Duistermaat-Heckman formula with application to circle actions and Poincare q-polynomials in twisted equivariant K-theory ' , Journal of Mathematical Physics , vol. 60 , no. 1 , 012302 . <https://doi.org/10.1063/1.5030105>

<http://hdl.handle.net/10138/311718>

<https://doi.org/10.1063/1.5030105>

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Cite as: J. Math. Phys. **60**, 012302 (2019); <https://doi.org/10.1063/1.5030105>

Submitted: 19 March 2018 . Accepted: 11 December 2018 . Published Online: 04 January 2019

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Cite as: J. Math. Phys. 60, 012302 (2019); doi: 10.1063/1.5030105

Submitted: 19 March 2018 • Accepted: 11 December 2018 •

Published Online: 4 January 2019



A. A. Bytsenko,^{1,a)} M. Chaichian,^{2,b)} and A. E. Gonçalves^{1,c)}

AFFILIATIONS

¹ Departamento de Física, Universidade Estadual de Londrina, Caixa Postal, 6001, Londrina-Paraná, Brazil

² Department of Physics, University of Helsinki, P.O. Box 64, 00014 Helsinki, Finland

^{a)} E-mail: aabyts@gmail.com

^{b)} E-mail: masud.chaichian@helsinki.fi

^{c)} E-mail: aedsongoncalves@gmail.com

ABSTRACT

In this paper, we deduce the sketch of proof of the Duistermaat-Heckman formula and investigate how the known Duistermaat-Heckman result could be specialized to the symplectic structure on the orbit space. The theorems of localization in equivariant cohomology not only provide us with beautiful mathematical formulas and stimulate achievements in algorithmic computations but also promote progress in theoretical and mathematical physics. We present the elliptic genera and the characteristic q -series for the circle actions and twisted equivariant K-theory, with the case of the symmetric group of n symbols separately analyzed. We show that the Poincaré q -polynomials admit presentation in terms of the Patterson-Selberg (or the Ruelle-type) spectral functions.

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I. INTRODUCTION

One of the aims of the present paper is to explain the sketch of the proof of the Duistermaat-Heckman theory.¹ Berline and Vergne showed how to derive the Duistermaat-Heckman formula from the localization theorem for equivariant cohomology (see, for example,² Theorem 7.19). The theorems of localization in equivariant cohomology not only provide beautiful mathematical formulas and stimulate achievements in algorithmic computations but also promote progress in theoretical physics.

One of the remarkable insights of the Duistermaat-Heckman formula is that we can deduce from it in a consistent way the Itzykson-Zuber formula for arbitrary compact connected Lie group action. The Itzykson-Zuber integration formula,³ for example, occurs essentially in matrix models (the Ising model on a random surface) where one considers the coupling of conformal matter to two-dimensional quantum gravity.⁴ This formula also appears in studies on higher-dimensional lattice gauge theories.

The specialization of the Duistermaat-Heckman formula leads to the Kirillov integral formula for irreducible representations of a compact connected Lie group,⁵ which has relevance for the geometric quantization: integration over the matrix groups G , which amounts to integration over G/T for T -invariant functions, has well-known importance in diverse areas such as integrable models, low-dimensional gauge theories, quantum gravity, and quantum chromodynamics.^{6,7}

Extension of the Duistermaat-Heckman formula to an infinite-dimensional manifold, the loop space of smooth maps from the circle S^1 to a compact manifold X , has been proposed for the first time by Witten. A formal application of the Duistermaat-Heckman theory to the partition function of $N = 1/2$ supersymmetric quantum mechanics reduces to a correct formula for the Dirac index.⁸ Subsequent arguments supporting these ideas (with strong mathematical base) were presented in Refs. 9 and 10.

Equivariant methods also can be used to define and analyze (twisted) K -theory.^{11–13} Applying techniques from the equivariant K -theory, one can construct a Chern character and K -theory Euler class. This manifests in the fact that the q -series elliptic genera can be expressed in terms of q -analogs of the classical special functions, in particular, the spectral Patterson-Selberg and Ruelle functions.¹⁴ In the present work, we would like to call the reader's attention to this connection by discussing some mathematical aspects associated with it.

The plan of this paper is as follows: In Sec. II, we begin with discussion of the Pfaffians. Note that particular attention should be paid to the choice of sign in the square root $\det^{1/2}(\xi)$, which depends on the choice of the orientation of a vector field ξ (we will frequently use the notation $\det^{1/2}(\xi) = \text{Pf}(\xi)$). We shall investigate the Hamiltonian G -space with attention to the symplectic structure on the orbit space, which is a necessary subject for quantization of the symplectic manifolds. Then we turn to the Duistermaat-Heckman formula.

In Sec. II A, we use the language of \mathcal{D} -modules—sheaves of modules over the sheaf of linear differential operators. That allows us to consider some applications of the localization formula and investigate how the known Riemann-Roch-Hirzebruch integral formula could be generalized to \mathcal{D} -modules.

Section III is devoted to the application to the one-dimensional setting of circle actions on a symplectic manifold. In Sec. III A, the developments of the direct analog of the K -theory Euler class make it possible to give a rigorous construction of the Chern character and the higher elliptic genera of level N in terms of q -series. We show that the q -series admit a presentation in the form of Ruelle-type spectral functions $\mathcal{R}(s)$. The symmetry properties (or more general the modular properties) of appropriate quantities can be analyzed by means of the functional equations for the spectral functions $\mathcal{R}(s)$. Results of this section are one of the novelties of the present work.

Finally, in Sec. IV, we discuss the twisted equivariant K -theory. The Poincaré polynomials are analyzed in Sec. IV A. Using the decomposition formula for the twisted K -theory, we discover the q -series for twisted symmetric products in terms of $\mathcal{R}(s)$ functions. Further on, we turn to the orbifold symmetric product in Sec. IV B. The orbifold Poincaré q -polynomials admit a presentation in terms of the Ruelle-type spectral functions.

II. Symplectic structure and the Duistermaat-Heckman formula

The Pfaffian. Suppose that manifold X is oriented, $\dim X = 2n$. Let $L_p(\xi)$ be a non-singular linear operator on $T_p X$, $\det L_p(\xi) \neq 0$, where ξ is a smooth vector field on X and $T_p X$ is the tangent space of X at the point p . Find an ordered orthonormal basis $\mathbf{e} = \mathbf{e}^{(p)} = \{e_j = e_j^{(p)}\}_{j=1}^{2n}$ of $T_p X$ such that

$$L_p(\xi)e_{2j-1} = \lambda_j e_{2j}, \quad L_p(\xi)e_{2j} = -\lambda_j e_{2j-1} \quad \text{for } 1 \leq j \leq n, \quad (2.1)$$

where each $\lambda_j \in \mathbb{R}/\{0\}$. Relative to \mathbf{e} , $L_p(\xi)$ is the skew-symmetric matrix. The Pfaffian $\text{Pf}_{\mathbf{e}}(L_p(\xi))$ of $L_p(\xi)$ associated to \mathbf{e} has the form² $\text{Pf}_{\mathbf{e}}(L_p(\xi))^2 = \det L_p(\xi)$.

Assume that G is a compact Lie group which acts smoothly on X (say on the left). Suppose that the metric $\langle \cdot, \cdot \rangle$ is G -invariant and \mathfrak{g} is the Lie algebra of G . For $\xi \in \mathfrak{g}$, there is an induced vector field $\xi^* \in V X$ (equal to the space of smooth vector fields on X). A vector field ξ^* is said to be non-degenerate if for every zero $p \in X$ of ξ^* the induced linear map $L_p(\xi^*): T_p X \rightarrow T_p X$ is non-singular. ξ^* is a Killing vector field and therefore $L_p(\xi^*)$ is skew-symmetric with respect to the inner product structure $\langle \cdot, \cdot \rangle_p$ on $T_p(X)$ and the non-singularity of $L_p(\xi^*)$ means that one can construct the square-root

$$(\det L_p(\xi^*))^{1/2} = (-1)^n \text{Pf}_{\mathbf{e}}(L_p(\xi^*)) = \prod_j \lambda_j. \quad (2.2)$$

Remark 2.1. The Euler number of an even-dimensional oriented manifold X with the Riemannian curvature R is given by the Gauss-Bonnet-Chern formula $\text{Eul}(X) = (2\pi)^{-n/2} \int_X \text{Pf}(-R)$. The reader can find the proof of this basic theorem of differential geometry, for example, in the excellent book², Section 1.6.

Symplectic structure on orbit space. Suppose X has a symplectic structure $\sigma, \sigma \in \Lambda^2 X$, which is a closed two-form, i.e., $d\sigma = 0$, such that for every $p \in X$ the corresponding skew-symmetric form $\sigma_p: T_p(X) \oplus T_p(X) \rightarrow \mathbb{R}$ is non-degenerate. For an integer $j \geq 0$, $\Lambda^j X$ denote the space of smooth complex differential forms of degree j on smooth manifold X . $d\xi: \Lambda^j X \rightarrow \Lambda^{j+1} X$ denotes the

exterior differentiation by ξ , while “ $\xi: \Lambda^j X \rightarrow \Lambda^{j-1} X$ ” denotes the interior differentiation by ξ . Suppose X is oriented by the Liouville form $\omega_\sigma = \frac{1}{n!} \sigma \wedge \cdots \wedge \sigma \in \Lambda^{2n} X / \{0\}$; suppose also that there is a map $\mathcal{H}: \mathfrak{g} \rightarrow C^\infty(X)$ which satisfies

$$\iota(\xi)\sigma + d\mathcal{H}(\xi) = 0, \quad \forall \xi \in \mathfrak{g}. \quad (2.3)$$

The existence of such a map \mathcal{H} is compatible with the assumption that the action of G on X is Hamiltonian. Let $\mathcal{H}VX$ denote the space of Hamiltonian vector fields on X (actually $\mathcal{H}VX$ is a Lie algebra).

The (left) action of G on X is called symplectic if $\xi \in \mathcal{H}VX$, $\forall \xi \in \mathfrak{g}$. The (left) action of G on X is called Hamiltonian if it is symplectic and if the Lie algebra homomorphism $\eta: \mathfrak{g} \rightarrow \mathcal{H}VX$ has a lift to $C^\infty(X)$. We note that such a \mathcal{H} will indeed satisfy condition (2.3). The triple (X, σ, \mathcal{H}) is called a Hamiltonian G -space.^{15,16}

Theorem 2.1 (see Ref. 17, Theorem 1). Assume that X and G are compact and the Riemannian metric \langle, \rangle on X is G -invariant; i.e., each $a \in G$ acts as an isometry of X . Assume now that the induced vector field ξ^* on X is non-degenerate; thus, the square-root in (2.2) is well-defined (and is non-zero) for $p \in X$, a zero of ξ^* (i.e., $\xi_p^* = 0$). Then for any cohomology class $[\tau] \in \Sigma(X, \xi^*, s)$, $s^{-1} = -2\pi i$, one has

$$\int_X [\tau] = (-1)^{n/2} \sum_{\substack{p \in X, \\ p = \text{a zero of } \xi^*}} \frac{p^*[\tau]}{[\det L_p(\xi^*)]^{1/2}}. \quad (2.4)$$

The Duistermaat-Heckman formula. We are now in the position to present the Duistermaat-Heckman formula in a form directly derivable from Theorem 2.1.

Theorem 2.2. Suppose that (X, σ, \mathcal{H}) is a Hamiltonian G -space, where G and X are compact and X is oriented by the Liouville form $\omega_\sigma = \frac{1}{n!} \sigma \wedge \cdots \wedge \sigma \in \Lambda^{2n} X / \{0\}$. For $c \in \mathbb{C}$ and for $\xi \in \mathfrak{g}$ with ξ^* non-degenerate, we have

$$\int_X e^{c\mathcal{H}(\xi)} \omega_\sigma = \left(\frac{2\pi}{c}\right)^n \sum_{\substack{p \in X \\ p = \text{a critical point of } \mathcal{H}(\xi)}} \frac{e^{c\mathcal{H}(\xi)(p)}}{[\det L_p(\xi)]^{1/2}}. \quad (2.5)$$

In addition, the critical points of $\mathcal{H}(\xi)$ are those where $d\mathcal{H}(\xi)$ vanishes.

Here some G -invariant Riemannian metric \langle, \rangle on X has been selected, and the square-root in (2.5) is as in (2.2).

Let us refer to some application of the Duistermaat-Heckman localization formula, a result when it is specialized, for example, to the symplectic structure on orbit space. In this connection, the basic example of a Hamiltonian G -space is that of an orbit \mathcal{O} in the dual space \mathfrak{g}^* of \mathfrak{g} under the co-adjoint action of G on \mathfrak{g}^* , which is induced by the adjoint action of G on \mathfrak{g} . In this construction, σ is chosen as the Kirillov symplectic form on $X = \mathcal{O}$ and \mathcal{H} is given by a canonical construction.

Of special interest is the case where \mathcal{O} is a quotient G/T of a compact, connected Lie group G modulo a maximal torus T . The Duistermaat-Heckman formula leads to the Kirillov integral formula for irreducible representations of G , which has relevance for the geometric quantization theory. Note that the classical studies on hypergeometric series have been initiated from the physics side; in particular, those studies are intimately related to the irreducible representations of the compact group $G = U(n)$ (see, for example, Refs. 18 and 19).

A. \mathcal{D} -modules and extension of the Riemann-Roch-Hirzebruch formula

Let $G_{\mathbb{C}}$ be a complex algebraic group with Lie algebra \mathfrak{g} . Suppose that $G_{\mathbb{C}}$ acts on a smooth manifold X and \mathcal{F} is a sheaf on X . A weak action of $G_{\mathbb{C}}$ on \mathcal{F} is an action of $G_{\mathbb{C}}$ on \mathcal{F} which extends the action of $G_{\mathbb{C}}$ on X .

Let \mathcal{D}_X be the sheaf of differential operators on X . In the case when \mathcal{F} is a \mathcal{D}_X -module, more structure has to be involved. Indeed, if φ denote the action of $G_{\mathbb{C}}$ on X , we get a morphism $d\varphi: \mathfrak{g} \rightarrow \Gamma(X, \mathcal{D}_X)$. $\Gamma(X, \mathcal{D}_X)$ denotes the global sections of \mathcal{D}_X , which means the ring of differential operators is defined on all of X . $\Gamma(X, \mathcal{D}_X)$ acts on itself by the commutator action ad , in such a manner that we get a map $ad \cdot d\varphi: \mathfrak{g} \rightarrow \text{End } \Gamma(X, \mathcal{D}_X)$. It is clear that \mathcal{D}_X has a weak action on $G_{\mathbb{C}}$, say γ , which yields the map $d\gamma: \mathfrak{g} \rightarrow \text{End } \Gamma(X, \mathcal{D}_X)$. Therefore, we can require the map $d\varphi$ to be $G_{\mathbb{C}}$ -equivariant and $ad \cdot d\varphi$ to coincide with $d\gamma$.

An interesting application of the localization formula is the generalization of the Riemann-Roch-Hirzebruch integral formula to \mathcal{D} -modules. Such a statement can be found in Ref. 20 where it uses the language of \mathcal{D} -modules—sheaves of modules over the sheaf of linear differential operators. Its flavour can be illustrated by the following example.

Let $G_{\mathbb{C}}$ be a connected complex algebraic linear reductive Lie group defined over \mathbb{R} and acting algebraically on a smooth complex projective variety X . Suppose that $G \subset G_{\mathbb{C}}$ is a real Lie subgroup lying between the group of real points $G_{\mathbb{C}}(\mathbb{R})$ and the identity component. Examine the sheaf of sections $\mathcal{O}(E)$ of a $G_{\mathbb{C}}$ -equivariant algebraic line bundle (E, ∇_E) over a $G_{\mathbb{C}}$ -invariant open algebraic subset $Y \subset X$ with a $G_{\mathbb{C}}$ -invariant algebraic flat connection ∇_E . Let $Y_{\mathbb{R}} \subset X$ be an open G -invariant subset (which may or may not be $G_{\mathbb{C}}$ -invariant). Consider the cohomology spaces $H^*(Y_{\mathbb{R}}, \mathcal{O}(E))$. The classical Riemann-Roch-Hirzebruch formula computes the index of E , i.e., the alternating sum

$$\sum_p (-1)^p \dim H^p(Y_{\mathbb{R}}, \mathcal{O}(E)) \quad (2.6)$$

with $Y_{\mathbb{R}} = Y = X$. For general $Y_{\mathbb{R}}$ and Y , these dimensions can be infinite. However, further work on this problem allows us to regard the vector spaces $H^*(Y_{\mathbb{R}}, \mathcal{O}(E))$ as the representations of G . Thus, as a substitute for the index, we can use for the character of the virtual representation the alternating sum (2.6). Note that for finite-dimensional representations the value of the character at the identity element $e \in G$ equals the dimension of the representation.

III. CIRCLE ACTIONS

First of all, it is convenient to begin with the one-dimensional setting of the circle actions on a symplectic manifold.

Recall that a symplectic manifold is a smooth manifold X equipped with a symplectic form σ , that is to say, a smooth anti-symmetric two-form $\sigma \in \Gamma(\wedge^2 TX^*)$ on X which is both non-degenerate (thus $\iota_{\xi} \sigma(x) \neq 0$, whenever ξ is a vector field that is non-vanishing) and closed (i.e., $d\xi = 0$). The symplectic manifolds are necessarily even-dimensional (because odd-dimensional anti-symmetric real matrices automatically have a zero eigenvalue and are thus degenerate). If X is a $2n$ -dimensional manifold, the Liouville measure on that manifold is defined as the volume form $\sigma^n/n!$, where, as before, we use σ^n to denote the n -fold wedge product of σ with itself, and we identify the volume forms with measures.

Given a smooth function $\mathcal{H} : X \rightarrow \mathbb{R}$ on a symplectic manifold (called the Hamiltonian), one can associate the Hamiltonian vector field $\xi = \xi_{\mathcal{H}} \in \Gamma(TX)$, defined by requiring that Eq. (2.3) holds. From the non-degeneracy of σ , we see that ξ vanishes precisely at the critical (or stationary) points of the Hamiltonian \mathcal{H} . We can exponentiate the Hamiltonian vector field to obtain a one-parameter group $\varphi(t): X \rightarrow X$ of smooth maps for $t \in \mathbb{R}$: $\varphi(t)x := e^{t\xi}x$. This is a smooth action of the additive group \mathbb{R} ; these maps are symplectomorphisms and, in particular, preserve Liouville measure. One can think of φ as a homomorphism from \mathbb{R} to the symplectomorphism group $\text{Symp}(X)$ of X .

A. Higher elliptic genera of level N and characteristic q -series

Let us consider the generalized elliptic genera for a manifold with an S^1 -action. Next, let us introduce the Ruelle-type spectral function of hyperbolic geometry $\mathcal{R}(s)$.¹⁴ The function $\mathcal{R}(s)$ is an alternating product of more complicated factors, each one being the so-called Patterson-Selberg zeta-function,

$$\prod_{n=\ell}^{\infty} (1 - q^{an+\varepsilon}) = \mathcal{R}(s = (a\ell + \varepsilon)(1 - i\varrho(\tau)) + 1 - a), \quad (3.1)$$

$$\prod_{n=\ell}^{\infty} (1 + q^{an+\varepsilon}) = \mathcal{R}(s = (a\ell + \varepsilon)(1 - i\varrho(\tau)) + 1 - a + i\sigma(\tau)), \quad (3.2)$$

where $q \equiv e^{2\pi i\tau}$, $\varrho(\tau) = \text{Re } \tau / \text{Im } \tau$, $\sigma(\tau) = (2 \text{Im } \tau)^{-1}$, a is a real number, $\varepsilon, b \in \mathbb{C}$, $\ell \in \mathbb{Z}_+$.

Let N be a fixed integer greater than 1 and x be a variable which is a complex number. Suppose that \mathfrak{h} is the upper half-plane of the complex numbers, $\tau \in \mathfrak{h}$. Let $L = 2\pi i(\mathbb{Z}\tau + \mathbb{Z})$ be a lattice and $\beta = 2\pi i(\frac{k}{N}\tau + \frac{l}{N})$, $0 \leq k < N$, $0 \leq l < N$, $\beta \neq 0$. It is useful to introduce the following function:²¹

$$\begin{aligned} \Phi(x) &= (1 - e^{-x}) \prod_{n=1}^{\infty} \frac{(1 - q^n e^{-x})(1 - q^n e^x)}{(1 - q^n)^2} \\ &= (1 - e^{-x}) \cdot \frac{\mathcal{R}(s = (1 + ix/2\pi\tau)(1 - i\varrho(\tau)))}{\mathcal{R}(s = 1 - i\varrho(\tau))} \\ &\quad \times \frac{\mathcal{R}(s = (1 - ix/2\pi\tau)(1 - i\varrho(\tau)))}{\mathcal{R}(s = 1 - i\varrho(\tau))}. \end{aligned} \quad (3.3)$$

In fact, Eq. (3.3) is a direct analog of the K-theory Euler class.²² The infinite product in Eq. (3.3) is similar to the inverse version of the Witten genus, which is an even function of the variable $z = i2\pi x$ and defines the cobordism invariance of oriented manifolds.

Also it admits a spin structure and takes values in $\mathbb{Z}[[q]]$. Let us take

$$f(x) := e^{(k/N)x} \Phi(x) \Phi(-\beta) / \Phi(x - \beta), \quad (3.4)$$

where the function $f(x)$ is elliptic with respect to a sublattice \tilde{L} of index N in L .²³ In the case of a compact almost complex manifold X , the total Chern class $C(TX)$ of X can be written formally as $C(TX) = \prod_{i=1}^d (1 + x_i)$. In this connection, the resulting Chern character takes the form

$$\begin{aligned} \text{Ch}\left(\bigotimes_{n \geq 1} \mathcal{S}_{q^n}(TX)\right)^{-1} &= \prod_j \prod_{n=1}^{\infty} (1 - q^n e^{x_j}) (1 - q^n e^{-x_j}) \\ &= \prod_j \mathcal{R}(s = (1 - ix_j/2\pi\tau)(1 - i\varrho(\tau))) \cdot \mathcal{R}(s = (1 + ix_j/2\pi\tau)(1 - i\varrho(\tau))). \end{aligned} \quad (3.5)$$

For a real vector bundle E over X , the symmetric powers of E become $\mathcal{S}_k(E) = \sum_{k \geq 0} \mathcal{S}^k(E) t^k$. The elliptic genus of level N can be defined as $\varphi_N(X) = \langle \prod_{j=1}^d \frac{x_j}{f(x_j)}, [X] \rangle$. The following assertion is known (see, for example, Ref. 22): if X has complex dimension m , then $\varphi_N(X)$ is a modular form of weight m under the actions of $SL(2, \mathbb{Z})$.

B. Properties of rigidity of elliptic operators

Recall that the elliptic genus has originated in order to find the generating series of rigid elliptic operators. Let as before X be a smooth compact manifold with an action of a group G . Suppose that \mathcal{D} is an elliptic operator on X (it commutes with the action). The kernel and the cokernel of \mathcal{D} are finite representations of G , and the Lefschetz number of \mathcal{D} at $g \in G$ (a character of G) can be written as follows: $\mathcal{L}_{\mathcal{D}}(g) = \text{tr}_g \ker \mathcal{D} - \text{tr}_g \text{coker } \mathcal{D}$. We presume that \mathcal{D} is rigid with respect to G if \mathcal{L} is independent of g ; in other words, we say that a character of G is constant. It is evident that in order to prove the rigidity of an elliptic operator with respect to a general compact connected Lie group action, we need to analyze its rigidity with respect to S^1 -action.

Remark 3.1. Let X be a spin manifold, and then the rigidity of the Dirac operator with respect to S^1 -action is the well known \hat{A} -vanishing theorem.²⁴ Using the two half-spinor representations of $\text{Spin}(2k)$, one can get two associated bundles on X denoted by $\{\Delta^+, \Delta^-\}$. For any real representation E of G , the Atiyah-Singer index theorem gives

$$\text{Index } D \otimes E = \int_X \text{ch } E \hat{A}(X), \quad (3.6)$$

where $D \otimes E$ is the twisted Dirac operator; $D \otimes E: \Gamma(\Delta^+ \otimes E) \rightarrow \Gamma(\Delta^- \otimes E)$. In addition, the index is defined to be $\text{Index } D \otimes E = \dim \ker D \otimes E - \dim \text{coker } D \otimes E$. The conjecture of the rigidity of $D \otimes TX$ and its further proof for the compact homogeneous spin manifolds have been given in Ref. 25.

The rigidity of the signature operator with respect to S^1 -action on the loop space has been conjectured by Witten; it is a highly non-trivial result. Motivated by quantum field theory; Witten has conjectured that the elliptic operators associated with the appropriate elliptic genera are rigid. Witten has also presented the rigidity conjectures for almost complex manifolds.

In many physical problems, the partition functions (and the Poincaré polynomials) are linked to the dimensions of appropriate homologies for topological spaces and admit infinite-product representations for the generating functions and the elliptic genera. In this connection, the symmetry modular properties of the q -deformed partition functions can be formulated by the specifics of Ramanujan's summation formula for the bilateral hypergeometric function. We define the general bilateral hypergeometric function ${}_r\psi_s$ as follows:²⁶

$${}_r\psi_s \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; q, z \right] = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_n}{(b_1, b_2, \dots, b_s; q)_n} (-1)^{(s-r)n} q^{(s-r)n(n-1)/2} z^n. \quad (3.7)$$

It is assumed that each term of this series is correctly defined.³¹ $(a)_n = a(a+1) \cdots (a+n-1)$. The shifted q -factorial has the form $(a; q)_n = 1$ for $n = 0$ and

$$\begin{aligned} (a; q)_n &= \frac{(a; q)_{\infty}}{(aq^n; q)_{\infty}} = \prod_{m=0}^{\infty} [(1 - aq^m)/(1 - aq^{m+n})] = (1-a)(1-aq) \cdots (1-aq^{n-1}), \\ n &= 1, 2, \dots, \end{aligned} \quad (3.8)$$

where n is a non-negative integer.

Extensive work in the theory of partition identities shows that the basic hypergeometric series provide the generating functions for numerous families of identities. Note that the symmetry properties (or in more general, the modular properties) for elliptic genera can be analyzed on the basis of (3.7) and also functional equations for the spectral functions of hyperbolic geometry $\mathcal{R}(s)$ [see, for example, Eq. (4.22) in Ref. 27].

IV. TWISTED EQUIVARIANT K-THEORY

Equivariant K-theory. At this point, we collect some basic properties of equivariant K-theory. Recall that (classically) topological K-theory asserts the association of any finite CW complex X to the category of vector bundles on X . Since it is an additive category, where all the exact sequences split, one can define the Grothendieck group $K(X)$ of this category. This can be achieved by taking the free Abelian group on all symbols $[E]$ for E , a vector bundle on X . $K(X)$ is a ring because one has a tensor product operation on the category of vector bundles (which commutes with direct sums). As a result, we can define a functor $K_G(\cdot)$ from compact G -spaces to Abelian groups, which descends to the homotopy category of equivariant spaces. (It is clear that in the non-equivariant case, $K_G(X)$ is a commutative ring.) With these preliminaries, recall the following definition: The equivariant K-group $K_G(X)$ of X is the Grothendieck group of equivariant vector bundles on X .

Some relations between the equivariant K-theory and the ordinary K-theory have to be mentioned: (i) There is a functor from equivariant vector bundles to vector bundles. (ii) There is a ring-homomorphism $K_G(X) \rightarrow K(X)$. Let $H \rightarrow G$ be a morphism of compact Lie groups, and then there is a map $K_G(X) \rightarrow K_H(X)$. (iii) Let X be a G -space and X/G be an ordinary space, and then there is a map $X \rightarrow X/G$. As a result, there is a map of a vector bundle on X/G to an ordinary vector bundle on X . (iv) There is a map $K(X/G) \rightarrow K_G(X)$ (note that if G acts freely on X , then the map $K(X/G) \rightarrow K_G(X)$ is an isomorphism).

Twisted version of equivariant K-theory. Let us discuss a twisted version of equivariant K-theory for a global quotient. Assume that G is a finite group, and suppose that for a given class $\alpha \in H^2(G, S^1)$ the compact Lie group extension $1 \rightarrow S^1 \rightarrow \tilde{G}_\alpha \rightarrow G \rightarrow 1$ is fulfilled. Here the group \tilde{G}_α is associated with the structure of a compact Lie group, where $S^1 \rightarrow \tilde{G}_\alpha$ is the inclusion of a closed subgroup. Let X be a finite G -CW complex.

We say that an α -twisted G -vector bundle on X is a complex vector bundle $E \rightarrow X$, if S^1 acts on the fibers through complex multiplication. In addition, the action extends to an action of \tilde{G}_α on E , which covers a given G -action on X . We denote the α -twisted G -equivariant K-theory of X by $K_G(X, \alpha)$, which is defined as the Grothendieck group of isomorphism classes of α -twisted G -bundles over X (see Ref. 12 for the precise definition). In the case $\alpha = 0$, the Lie group extension corresponds to the split extension $G \times S^1$. In addition, any G -vector bundle can be turned into a $G \times S^1$ -bundle by means of a scalar multiplication on the fibers. It means that a $G \times S^1$ -bundle restricts to a G -bundle. As a result, we have $K_G^*(X, \alpha = 0) = K_G^*(X)$.

For a finite group G and G -CW complex X , decomposition for K-theory takes the form¹¹

$$K_G^*(X, \alpha) \otimes \mathbb{C} \cong \bigoplus_{\{(g)|g \in G\}} (K^*(X^{(g)}) \otimes L_g^\alpha)^{Z_G(g)}. \quad (4.1)$$

In Eq. (4.1), (g) is the conjugacy class of $g \in G$, $Z_G(g)$, and denotes the centralizer of g in G , while L_g^α denotes the character for the centralizer $Z_G(g)$.

A. The symmetric product

Let G be a finite group and X be a G -space, on which a normal subgroup A acts trivially. The G -equivariant K-theory of X can be decomposed as a direct sum of twisted equivariant K-theories of X parametrized by the orbits of the conjugation action of G on the irreducible representations of A ²⁸ (Theorems 3.2 and 3.4).

Now we turn to the symmetric group and denote by $G = \mathfrak{S}_n$ the symmetric group on n symbols. As before, we denote the non-trivial class by α . Using the decomposition formula (Theorem 3.4 of Ref. 28), one can calculate $K_{\mathfrak{S}_n}^*(X^n, \alpha)$, where the group acts on the n -fold product of a manifold X by permutation of coordinates. At last, the formula for the twisted symmetric products takes the form

$$\begin{aligned} \sum q^n \chi(K_{\mathfrak{S}_n}^*(X^n, \alpha) \otimes \mathbb{C}) &= \prod_{n>0} (1 - q^{2n-1})^{-\chi(X)} + \prod_{n>0} (1 + q^{2n-1})^{\chi(X)} \\ &\times [1 + \frac{1}{2} \prod_{n>0} (1 + q^{2n})^{\chi(X)} - \frac{1}{2} \prod_{n>0} (1 - q^{2n})^{\chi(X)}]. \end{aligned} \quad (4.2)$$

In Eq. (4.2), $\chi(X)$ is the Euler characteristic of X . In terms of spectral functions $\mathcal{R}(s)$, Eq. (4.2) acquires the form

$$\sum q^n \chi(K_{\mathfrak{S}_n}^*(X^n, \alpha) \otimes \mathbb{C}) = \mathcal{R}(s = -i\varrho(\tau))^{-\chi(X)} + \mathcal{R}(s = -i\varrho(\tau) + i\sigma(\tau))^{\chi(X)} \\ \times [1 + \frac{1}{2} \mathcal{R}(s = 1 - 2i\varrho(\tau) + i\sigma(\tau))^{\chi(X)} - \frac{1}{2} \mathcal{R}(s = 1 - 2i\varrho(\tau))^{\chi(X)}]. \quad (4.3)$$

B. The orbifold symmetric product

We will analyze the Poincaré polynomial of orbispace $[X^n/\mathfrak{S}_n] := [X \times \cdots \times X/\mathfrak{S}_n]$,²⁹ whose category objects are n -tuples (x_1, \dots, x_n) of points in X . In addition, the arrows are elements of the form $(x_1, \dots, x_n; \sigma)$, where $\sigma \in \mathfrak{S}_n$. $(x_1, \dots, x_n; \sigma)$ has its source as (x_1, \dots, x_n) while its target as $(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ [the mentioned category is a groupoid for the inverse of $(x_1, \dots, x_n; \sigma)$, which is $(x_1, \dots, x_n; \sigma^{-1})$. This is the reason why we can consider $[X^n/\mathfrak{S}_n]$ as an orbispace].

Poincaré q -polynomials. Let X be a topological space, denoted by $P(X, y)$, its Poincaré polynomial, $P(X, y) = \sum_j b^j(X) y^j$. Here $b^j(X)$ is the j th Betti number of X . The following formula has been proved by Macdonald:³⁰

$$\sum_{n=0}^{\infty} P(X^n/\mathfrak{S}_n, y) q^n = \frac{\prod_j (1 + q y^{2j+1})^{b^{2j+1}(X)}}{\prod_j (1 - q y^{2j})^{b^{2j}(X)}}. \quad (4.4)$$

For $y = -1$, Eq. (4.4) reduces to the formula for the Euler characteristic of the symmetric product $\sum_{n=0}^{\infty} \chi(X^n/\mathfrak{S}_n) q^n = (1 - q)^{-\chi(X)}$, which is valid for the topological space X whose cohomology $H^j(X; \text{real})$ is finitely generated for each $j \geq 0$. Using the \mathfrak{S}_n -equivariant K-theory of X^n , one can arrive at the following formula for generation functions:²⁹ $\sum_{n=0}^{\infty} \chi_{\mathfrak{S}_n}(X^n/\mathfrak{S}_n) q^n = \prod_{j>0} (1 - q^j)^{-\chi(X)}$. In terms of the spectral functions $\mathcal{R}(s)$, Eq. (4.4) acquires the form

$$\sum_{n=0}^{\infty} P(X^n/\mathfrak{S}_n, y) q^n = \frac{\mathcal{R}(s = (1 - i(\log(y))(2j+1)/(2\pi\tau))(1 - i\varrho(\tau)) + i\sigma(\tau))^{b_{2j+1}(X)}}{\mathcal{R}(s = (1 - i(\log(y))(j/\pi\tau))(1 - i\varrho(\tau))^{b_{2j}(X)}}. \quad (4.5)$$

The Poincaré orbifold polynomial can be defined as follows:²⁹

$$P_{\text{orb}}([X/G], y) := \sum y^j \text{rank } H_{\text{orb}}^j([X/G]; \text{real}) \equiv \sum y^j b_{\text{orb}}^j([X/G]), \quad (4.6)$$

where b_{orb}^j is the j th orbifold Betti number.

Remark 4.1. For any arbitrary (fixed) positive integers m_1, \dots, m_r , the following equality holds:

$$\sum_{n_1, \dots, n_r=0}^{\infty} F(n_1, \dots, n_r) z_1^{n_1}, \dots, z_r^{n_r} = \sum_{N=0}^{\infty} \sum_{\substack{n_1 m_1 + \dots + n_r m_r = N \\ n_1, \dots, n_r \geq 0}} F(n_1, \dots, n_r) z_1^{n_1}, \dots, z_r^{n_r}. \quad (4.7)$$

For the symmetric product, viewed as an orbifold groupoid $[X^n/\mathfrak{S}_n]$, it follows that

$$H_{\text{orb}}^*([X^n/\mathfrak{S}_n]; \text{real}) \cong \bigoplus_{\sum j n_j = n} \bigotimes_j H^*(X^{n_j}; \text{real})^{\mathfrak{S}_{n_j}}. \quad (4.8)$$

Using Eq. (4.7) for calculating the orbifold Poincaré polynomial, we obtain

$$\sum_{n=0}^{\infty} q^n P_{\text{orb}}([X^n/\mathfrak{S}_n], y) = \sum_{n=0}^{\infty} q^n \left(\sum_{\sum j n_j = n} \prod_j P(X^{n_j}/\mathfrak{S}_{n_j}, y) \right) \\ = \prod_{j>0} \prod_{n \geq 0} \frac{(1 + q^n y^{2j+1})^{b^{2j+1}(X)}}{(1 - q^n y^{2j})^{b^{2j}(X)}} \\ = \prod_{j>0} \frac{[\mathcal{R}(s = -i(2j+1)\log(y)/(2\pi\tau)(1 - i\varrho(\tau)) + i\sigma(\tau))]^{b_{2j+1}(X)}}{[\mathcal{R}(s = -ij\log(y)/(\pi\tau)(1 - i\varrho(\tau))]^{b_{2j}(X)}}. \quad (4.9)$$

For these formulas to be valid, the cohomology of X must be finitely generated at each n . The first line in (4.9) is similar to the equality (2.32) of Ref. 29. As before, $b^j(X)$ is the j th Betti number of X .

ACKNOWLEDGMENTS

We are grateful to Anca Tureanu for many discussions and several suggestions. A.A.B. and A.E.G. would like to acknowledge the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq, Brazil) and Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES, Brazil) for financial support.

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- ³¹As an example, consider the ${}_3\psi_2$ -summation formula.²⁶ Let n be a non-negative integer; the q -analog of the Pfaff-Saalschütz summation formula for ${}_3\psi_2$ is

$${}_3\psi_2 \left[\begin{matrix} a, b, q^{-n}; q, q \\ c, abq^{1-n}/c \end{matrix} \right] = \frac{(c/a; q)_n (c/b; q)_n}{(c; q)_n (c/ab; q)_n}.$$